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#### Linear Quantization by Effective Resistance Sampling

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Joint work with Aarti Singh

- \* The linear model:  $y = X\beta_0$ 
  - \* X: n by p "design" matrix, with full knowledge
  - \* *y*: *n*-dim vector, the sensing result
  - \*  $\beta_0$ : p-dim unknown signal to be recovered

- \* The linear model:  $y = X\beta_0$
- \* The quantized sensing problem:
  - \* Measurements of *y* cannot be made in arbitrary precision
  - \* A total of k bits allocated to each measurement  $y_i$
  - \* Each  $y_i$  rounded to the nearest integer with  $k_i$  binary bits.

$$\widetilde{y}_i = 2^{-(k_i - 1)} \cdot \text{round} \left[ 2^{k_i - 1} \frac{y_i}{M} \right]$$

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Range of y

- \* The linear model:  $y = X\beta_0$
- \* The quantized sensing problem:
  - \* Measurements of *y* cannot be made in arbitrary precision
- \* Example applications:
  - \* Brain activity measurements: total signal strength limited
  - \* Distributed sensing: signal communication limited

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\* Question: how to allocate measurement bits to achieve the best statistical efficiency?

#### DITHERING

- \* "Dithering":  $\widetilde{y}_i = 2^{-(k_i 1)} \cdot \text{round} \left[ 2^{k_i 1} \left( \frac{y_i}{M} + \delta_i \right) \right]$ 
  - \* Introducing artificial noise for independent statistical error
  - \* Equivalent model:  $\widetilde{y}_i = \langle x_i, \beta_0 \rangle + \varepsilon_i$

$$\mathbb{E}[\varepsilon_i] = 0 \qquad \qquad \mathbb{E}[\varepsilon_i^2] \le 4^{-(k_i+1)} M^2$$

#### DITHERING

uniform noise between two values

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Weighted Ordinary Least Squares (OLS)

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$$W = \operatorname{diag}(w_1, w_2, \cdots, w_n)$$

$$= \operatorname{diag}(4^{k_1+1}, 4^{k_2+1}, \cdots, 4^{k_n+1})$$

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$$\mathbb{E}\|\widehat{\beta}_{\mathbf{k}} - \beta_0\|_2^2 \le M^2 \cdot \operatorname{tr}\left[\sum_{i=1}^n 4^{k_i+1}x_ix_i^{\top}\right]^{-1}$$

\* Optimal quantization:

$$\min_{\mathbf{k}} \operatorname{tr}[X^{\top} W X]^{-1} \quad s.t. \quad k_1 + \dots + k_n \le k, \quad k_i \in \mathbb{N}$$

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Combinatorial... hard!

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Continuously relaxed optimal quantization:

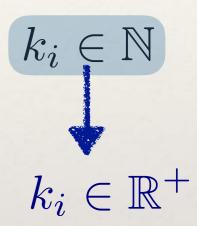
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\* A re-formulation:

$$\min \operatorname{tr} \left[ \sum_{i=1}^{n} 4^{k_i + 1} x_i x_i^{\top} \right]^{-1} \quad s.t. \quad \sum_{i=1}^{n} k_i \le k$$

$$\min \operatorname{tr} \left[ \sum_{i=1}^{n} w_i x_i x_i^{\top} \right]^{-1} \quad s.t. \quad \sum_{i=1}^{n} \log_4(w_i) - 1 \le k$$

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 ex objective

$$s.t. \sum_{i=1}^{n} k_i \le k$$

Convex objective

min 
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  $s.t.$   $\sum_{i=1}^n \log_4(w_i) - 1 \le k$ 

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$$\min \operatorname{tr} \left[ \sum_{i=1}^{n} w_i x_i x_i^{\top} \right]^{-1} + \lambda \left[ \sum_{i=1}^{n} \log_4(w_i) - (n-k) \right]$$

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convex objective

concave objective

$$\min \operatorname{tr} \left[ \sum_{i=1}^{n} w_i x_i x_i^{\top} \right]^{-1} + \lambda \left[ \sum_{i=1}^{n} \log_4(w_i) - (n-k) \right]$$

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- \* A re-formulation:
  - \* DC (Difference of Convex functions) programming:

$$\min \operatorname{tr} \left[ \sum_{i=1}^{n} w_i x_i x_i^{\top} \right]^{-1} - \lambda \left[ -\sum_{i=1}^{n} \log_4(w_i) + (n-k) \right]$$

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- \* How to obtain integral solutions? "Sparsify" k
  - \* Idea 1: round to the nearest integer
  - \* Problem: might cause objective to increase significantly

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- \* How to obtain integral solutions? "Sparsify" k
  - \* Idea 2: **simple** sampling
    - \* Sample i from the distribution normalized by **k**
    - \* k(i) = k(i) + 1
  - \* Problem: slow convergence (require large budget *k*)

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  - \* Idea 3: **effective resistance** sampling

$$t \sim p_t \propto 4^{k_t + 1} \ell_t$$

\* Advantage: fast convergence (*k* independent of condition numbers of *X* or *W*.

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- \* How to obtain integral solutions? "Sparsify" k
  - \* Idea 3: **effective resistance** sampling

$$t \sim p_t \propto 4^{k_t+1} \ell_t \quad \begin{array}{l} \text{Effective resistance:} \\ \ell_t = x_t^\top [W^*]^{-1} x_t \end{array}$$

\* Advantage: fast convergence (*k* independent of condition numbers of *X* or *W*.

# OPEN QUESTIONS

\* Most important question: how to solve (continuous)

$$\min \operatorname{tr} \left[ \sum_{i=1}^{n} 4^{k_i + 1} x_i x_i^{\top} \right]^{-1} \qquad s.t. \quad \sum_{i=1}^{n} k_i \le k$$

- \* Some ideas:
  - \* Is the objective quasi-convex or directional convex?
  - \* Are local minima also global, or approximately global?
    - Escaping saddle point methods?
  - \* Are there adequate **convex** relaxations?

Thank you! Questions